NEAPIBRĖŽTINIAI INTEGRALAI

INTEGRAVIMO METODAI II

<u>Racionaliųjų funkcijų integravimas</u>

1. $\int \frac{(x+2)\cdot dx}{x^3-2x^2} = \int \frac{(x+2)\cdot dx}{x^2\cdot(x-2)}$

Išskaidome racionaliąją trupmeną ir apskaičiuojame nežinomus koeficientus:

$$\frac{x+2}{x^2\cdot(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

Subendravardiklinamos trupmenos ir sulyginami trupmenų skaitikliai:

$$A\cdot x\cdot(x-2)+B\cdot(x-2)+C\cdot x^2 = x+2$$
$$(A+C)\cdot x^2+(-2A+B)\cdot x-2B = x+2$$

Koeficientai prie vienodų x laipsnio rodiklių:

$$\begin{cases} x^2 : A+C=0 \\ x^1 : -2A+B=1 \\ x^0 : -2B=2 \end{cases} \Rightarrow \begin{cases} C=-A=1 \\ A=\frac{B-1}{2}=\frac{-1-1}{2}=-1 \\ B=\frac{2}{-2}=-1 \end{cases}$$

Tada turime:

$$\int \frac{(x+2)\cdot dx}{x^3-2x^2} = \int \frac{(x+2)\cdot dx}{x^2\cdot(x-2)} = \int\left(\frac{-1}{x}+\frac{-1}{x^2}+\frac{1}{x-2}\right)\cdot dx = -\ln|x|+\frac{1}{x}+\ln|x-2|+C.$$

2. $\int \frac{(11x+16)\cdot dx}{(x-1)\cdot(x+2)^2}$

Išskaidome racionaliąją trupmeną ir apskaičiuojame nežinomus koeficientus:

$$\frac{11x+16}{(x-1)\cdot(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

Subendravardiklinamos trupmenos ir sulyginami trupmenų skaitikliai:

$$A\cdot(x+2)^2+B\cdot(x+2)\cdot(x-1)+C\cdot(x-1)=11x+16;$$
$$A\cdot(x^2+4x+4)+B\cdot(x^2+x-2)+C\cdot(x-1)=11x+16;$$
$$(A+B)\cdot x^2+(4A+B+C)\cdot x+4A-2B-C=11x+16.$$

Koeficientai prie vienodų x laipsnio rodiklių:

$$\begin{cases} x^2: A+B=0 \\ x^1: 4A+B+C=11 \\ x^0: 4A-2B-C=16 \end{cases}$$

iš (1): $A=-B$

iš (3): $C=4A-2B-16=-4B-2B-16=-6B-16$

iš (2): $4\cdot(-B)+B+(-6B-16)=11 \Rightarrow B=\frac{27}{-9}=-3$

$A=-B=3$

$C=-6B-16=-6\cdot(-3)-16=2$

Tada turime:

$$\int\frac{(11x+16)\cdot dx}{(x-1)\cdot(x+2)^2}=\int\left(\frac{3}{x-1}+\frac{-3}{x+2}+\frac{2}{(x+2)^2}\right)\cdot dx=3\cdot\ln|x-1|-3\cdot\ln|x+2|-\frac{2}{x+2}+C.$$

3. $\int\frac{x^3-2x^2+3x+2}{x^3-3x^2+2x}\cdot dx$

Išskiriame trupmenos sveikąją dalį:

$$\begin{array}{l|l} \;x^3-2x^2+3x+2 & x^3-3x^2+2x \\ \cline{2-2} -\;x^3-3x^2+2x & 1 \\ \hline \;x^2+x+2 & \end{array}$$

$$\int\frac{x^3-2x^2+3x+2}{x^3-3x^2+2x}\cdot dx=\int\left(1+\frac{x^2+x+2}{x^3-3x^2+2x}\right)\cdot dx=\int dx+\int\frac{x^2+x+2}{x^3-3x^2+2x}\cdot dx=x+\int\frac{x^2+x+2}{x^3-3x^2+2x}\cdot dx$$

Išskaidome racionaliąją trupmeną ir apskaičiuojame nežinomus koeficientus:

$$\frac{x^2+x+2}{x^3-3x^2+2x}=\frac{x^2+x+2}{x\cdot(x^2-3x+2)}=\frac{x^2+x+2}{x\cdot(x-1)\cdot(x-2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-2}$$

Subendravardiklinamos trupmenos ir sulyginami trupmenų skaitikliai:

$$A\cdot(x-1)\cdot(x-2)+B\cdot x\cdot(x-2)+C\cdot x\cdot(x-1)=x^2+x+2$$
$$A\cdot(x^2-3x+2)+B\cdot(x^2-2x)+C\cdot(x^2-x)=x^2+x+2$$
$$(A+B+C)\cdot x^2+(-3A-2B-C)\cdot x+2A=x^2+x+2$$

Koeficientai prie vienodų x laipsnio rodiklių:

$$\begin{cases} x^2: \; A+B+C=1 \\ x^1: \; -3A-2B-C=1 \\ x^0: \; 2A=2 \end{cases}$$

$$A=\frac{2}{2}=1$$

$$\begin{cases} C=1-A-B \\ C=-3A-2B-1 \end{cases} \Rightarrow 1-A-B=-3A-2B-1 \; \Rightarrow \; B=-2A-2=-2\cdot 1-2=-4$$

$$C=1-1-(-4)=4$$

Tada turime:

$$\int \frac{x^2+x+2}{x^3-3x^2+2x}\cdot dx=\int\left(\frac{1}{x}+\frac{-4}{x-1}+\frac{4}{x-2}\right)\cdot dx=\ln|x|-4\ln|x-1|+4\ln|x-2|+C$$

$$\Rightarrow \int \frac{x^3-2x^2+3x+2}{x^3-3x^2+2x}\cdot dx=x+\ln|x|-4\ln|x-1|+4\ln|x-2|+C.$$

4.

$$\int \frac{x^2\cdot dx}{\left(x^2+1\right)^2}=\frac{1}{2}\cdot\int\frac{x\cdot d\left(x^2\right)}{\left(x^2+1\right)^2}=\frac{1}{2}\cdot\int\frac{x\cdot d\left(x^2+1\right)}{\left(x^2+1\right)^2}=-\frac{1}{2}\cdot\int x\cdot d\left(\frac{1}{x^2+1}\right)=-\frac{1}{2}\cdot x\cdot\frac{1}{x^2+1}+\frac{1}{2}\cdot\int\frac{1}{x^2+1}\cdot dx=$$

$$=-\frac{1}{2}\cdot x\cdot\frac{1}{x^2+1}+\frac{1}{2}\cdot arctgx+C$$

Irracionaliųjų funkcijų integravimas

5. $$\int \frac{dx}{(2-x)\sqrt{1-x}} = \begin{bmatrix} \sqrt{1-x}=t \\ x = 1-t^2 \\ dx = -2\cdot t\cdot dt \end{bmatrix} = \int \frac{-2\cdot t\cdot dt}{(2-1+t^2)\cdot t} = -2\cdot\int\frac{dt}{t^2+1} = -2\cdot arctgt + C = -2\cdot arctg\sqrt{1-x}+C$$

6. $$\int \frac{dx}{\sqrt{x^2+x-1}} = \int \frac{d\left(x+\frac{1}{2}\right)}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}} = \ln\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\right|+C = \ln\left|x+\frac{1}{2}+\sqrt{x^2+x-1}\right|+C$$

7.

$$\int \frac{x^2+\sqrt{1+x}}{\sqrt[3]{1+x}}\cdot dx = \begin{bmatrix} \sqrt[6]{1+x}=t \\ x=t^6-1 \\ dx = 6\cdot t^5\cdot dt \end{bmatrix} = \int \frac{(t^6-1)^2+t^3}{t^2}\cdot 6\cdot t^5\cdot dt = 6\cdot\int\left((t^6-1)^2+t^3\right)\cdot t^3\cdot dt =$$

$$= 6\cdot\int(t^{12}-2t^6+1+t^3)\cdot t^3\cdot dt = 6\cdot\int(t^{15}-2t^9+t^6+t^3)\cdot dt = 6\cdot\left(\frac{1}{16}\cdot t^{16}-2\cdot\frac{1}{10}\cdot t^{10}+\frac{1}{7}\cdot t^7+\frac{1}{4}\cdot t^4\right)+C$$

$$= 6\cdot\left(\frac{1}{16}\cdot\sqrt[6]{1+x}^{16}-2\cdot\frac{1}{10}\cdot\sqrt[6]{1+x}^{10}+\frac{1}{7}\cdot\sqrt[6]{1+x}^{7}+\frac{1}{4}\cdot\sqrt[6]{1+x}^{4}\right)+C=$$

$$= 6\cdot\left(\frac{1}{16}\cdot(1+x)^{\frac{8}{3}}-\frac{1}{5}\cdot(1+x)^{\frac{5}{3}}+\frac{1}{7}\cdot(1+x)^{\frac{7}{6}}+\frac{1}{4}\cdot(1+x)^{\frac{2}{3}}\right)+C$$

8.

$$\int \frac{dx}{\sqrt[3]{x^2}+3\sqrt{x}} = \begin{bmatrix} \sqrt[6]{x}=t \\ x=t^6 \\ dx=6t^5\cdot dt \end{bmatrix} = \int\frac{6t^5\cdot dt}{t^4+3t^3} = \int\frac{6t^2\cdot dt}{t+3}$$

Išskiriame sveikąją dalį:

$$\begin{array}{rl|l} & 6t^2 & t+3 \\ - & 6t^2+18t & 6t-18 \\ \hline & -18t & \\ - & -18t-54 & \\ \hline & 54 & \end{array}$$

$$\int\frac{6t^2\cdot dt}{t+3} = \int\left(6t-18+\frac{54}{t+3}\right)\cdot dt = \int 6t\cdot dt - \int 18\cdot dt + \int\frac{54}{t+3}\cdot dt = 3t^2-18t+54\cdot\ln|t+3|+C.$$

9.

$$\int \frac{dx}{x^2 \cdot \sqrt{4x^2-5}}$$

$$\frac{1}{x^2 \cdot \sqrt{4x^2-5}} = x^{-2} \cdot \left(4x^2-5\right)^{-\frac{1}{2}}$$

$$x^m \cdot \left(a+bx^n\right)^p : m=-2,\ n=2,\ p=-\frac{1}{2}$$

$$\frac{m+1}{n}+p=\frac{-2+1}{2}-\frac{1}{2}=-\frac{1}{2}-\frac{1}{2}=-1$$ - sveikas skaičius

Taikome keitinį:

$$\left[\begin{array}{l} t^2=b+\frac{a}{x^n}=4+\frac{-5}{x^2};\quad x^2=\frac{5}{4-t^2};\quad x=\sqrt{\frac{5}{4-t^2}} \\ \sqrt{4x^2-5}=\sqrt{4\cdot\frac{5}{4-t^2}-5}=\sqrt{\frac{20-20+5t^2}{4-t^2}}=\frac{\sqrt{5}t}{\sqrt{4-t^2}} \\ dx=\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{5}{4-t^2}}}\cdot\left(-\frac{5}{\left(4-t^2\right)^2}\cdot(-2t)\right)\cdot dt=\frac{\sqrt{5}t}{\left(4-t^2\right)^{\frac{3}{2}}}\cdot dt \end{array}\right]$$

Tada turime:

$$\int \frac{dx}{x^2 \cdot \sqrt{4x^2-5}} = \int \frac{\frac{\sqrt{5}t}{\left(4-t^2\right)^{\frac{3}{2}}}\cdot dt}{\frac{5}{4-t^2}\cdot\frac{\sqrt{5}t}{\sqrt{4-t^2}}} = \frac{1}{5}\cdot\int dt = \frac{1}{5}\cdot t + C = \frac{1}{5}\cdot\sqrt{4-\frac{5}{x^2}}+C\,.$$